Chapter 10

Rotation

10.2 The Rotational Variables

A **rigid body** is a body that can rotate with all its parts locked together and without any change in its shape.

A **fixed axis** means that the rotation occurs about an axis that does not move.

Figure skater Sasha Cohen in motion of pure rotation about a vertical axis. *(Elsa/Getty Images, Inc.)*

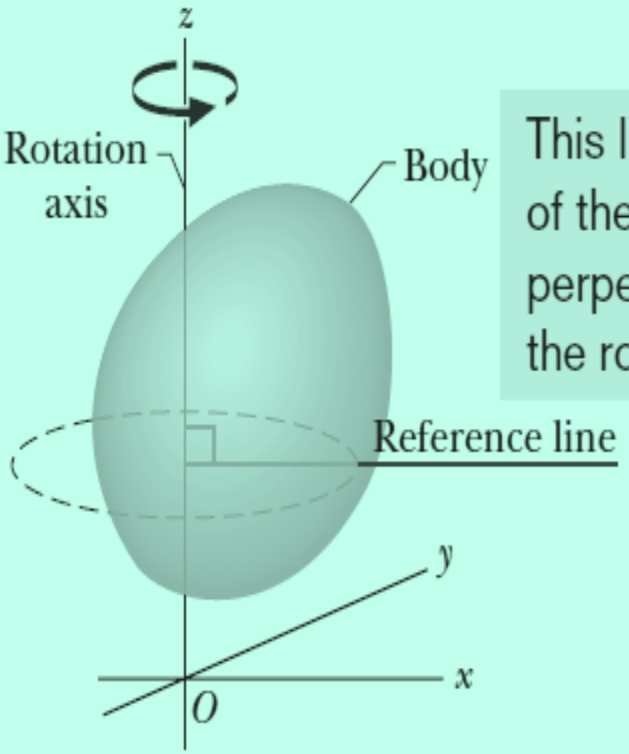


Fig. 10-2 A rigid body of arbitrary shape in pure rotation about the z axis of a coordinate system. The position of the *reference line* with respect to the rigid body is arbitrary, but it is perpendicular to the rotation axis. It is fixed in the body and rotates with the body.

10.2 The Rotational Variables: Angular Position

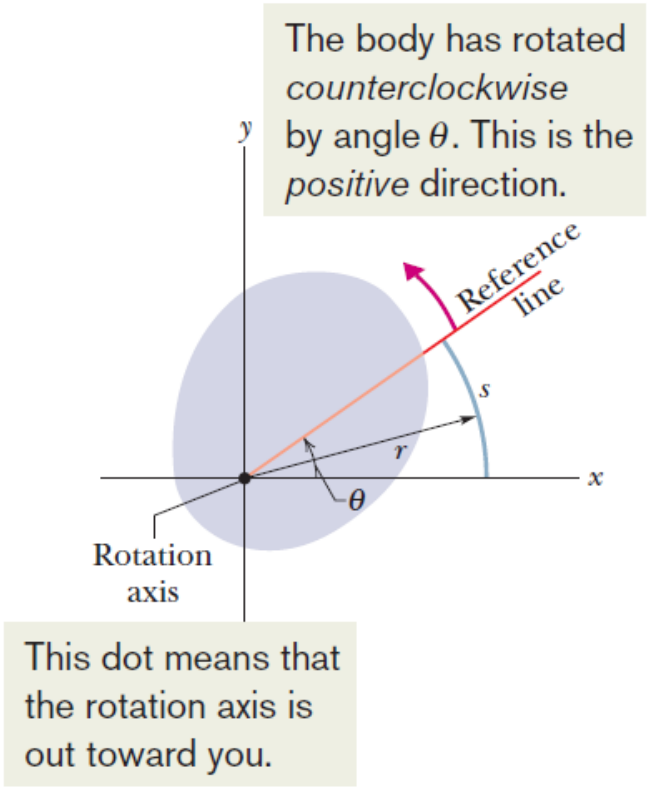


Fig. 10-3 The rotating rigid body of Fig. 10-2 in cross section, viewed from above. The plane of the cross section is perpendicular to the rotation axis, which now extends out of the page, toward you. In this position of the body, the reference line makes an angle θ with the x axis.

$$\theta = \frac{s}{r} \qquad \text{(radian measure)}.$$

Here s is the length of a circular arc that extends from the x axis (the zero angular position) to the reference line, and *r* is the radius of the circle. An angle defined in this way is measured in **radians (rad).**

$$1 \text{ rev} = 360° = \frac{2\pi r}{r} = 2\pi \text{ rad},$$

$$1 \text{ rad} = 57.3° = 0.159 \text{ rev}.$$

10.2 The Rotational Variables: Angular Displacement

- If a body rotates about the rotation axis as in Fig. 10-4, changing the angular position of the reference line from Θ_1 to Θ_2, the body undergoes an angular displacement $\Delta\Theta$ given by:

$$\Delta\theta = \theta_2 - \theta_1.$$

- ***An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.***

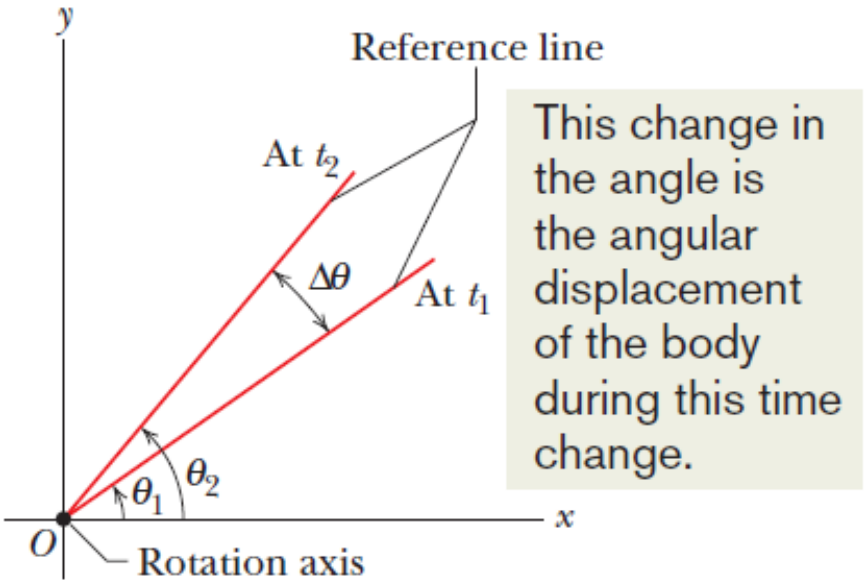


Fig. 10-4 The reference line of the rigid body of Figs. 10-2 and 10-3 is at angular position θ_1 at time t_1 and at angular position θ_2 at a later time t_2. The quantity $\Delta\theta\,(=\theta_2-\theta_1)$ is the angular displacement that occurs during the interval $\Delta t\,(=t_2-t_1)$. The body itself is not shown.

10.2 The Rotational Variables: Angular Velocity

- Suppose that our rotating body is at angular position Θ_1 at time t_1 and at angular position Θ_2 at time t_2. Then the average angular velocity of the body in the time interval t from t_1 to t_2 is defined to be:

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t},$$

- The instantaneous angular velocity ω is the limit of the ratio as Δt approaches zero.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

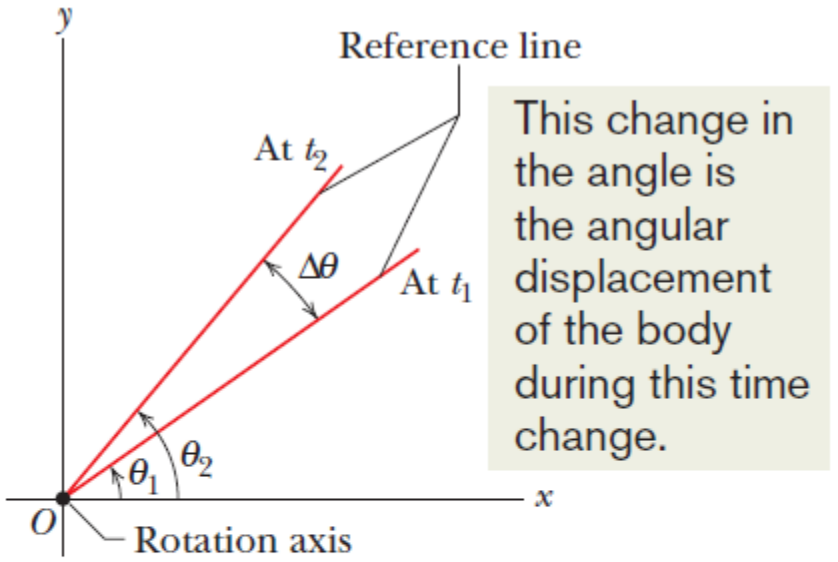


Fig. 10-4 The reference line of the rigid body of Figs. 10-2 and 10-3 is at angular position θ_1 at time t_1 and at angular position θ_2 at a later time t_2. The quantity $\Delta\theta\,(= \theta_2 - \theta_1)$ is the angular displacement that occurs during the interval $\Delta t\,(= t_2 - t_1)$. The body itself is not shown.

10.2 The Rotational Variables: Angular Acceleration

- If the angular velocity of a rotating body is not constant, then the body has an angular acceleration.

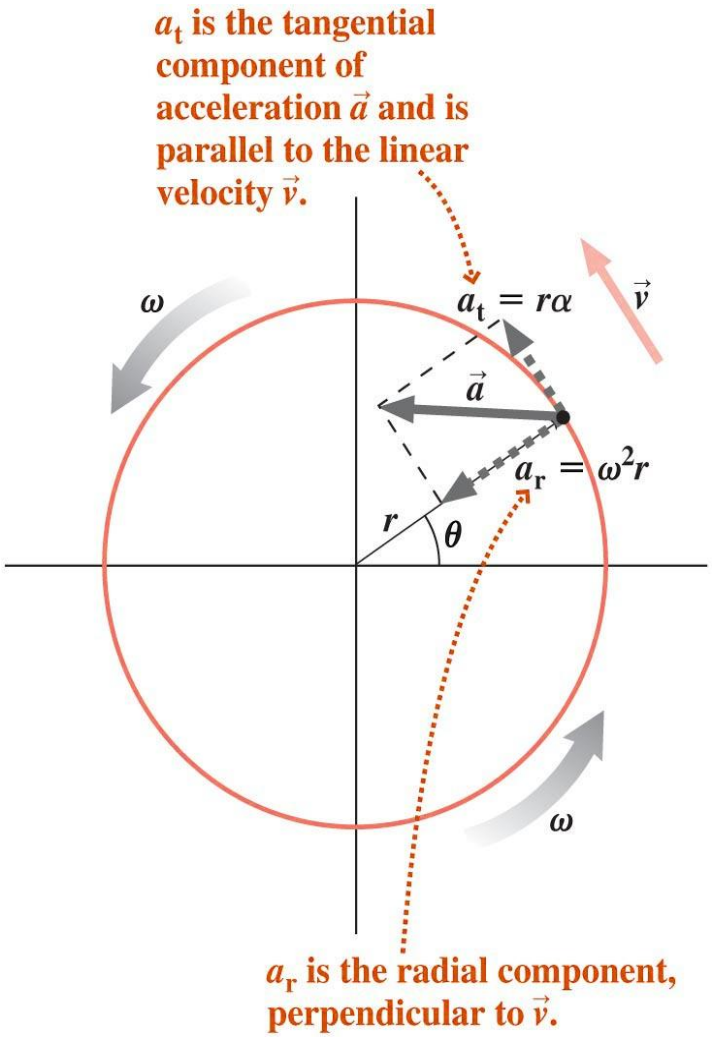
- If ω_2 and ω_1 are the angular velocities at times t_2 and t_1, respectively, then the average angular acceleration of the rotating body in the interval from t_1 to t_2 is defined as:

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t},$$

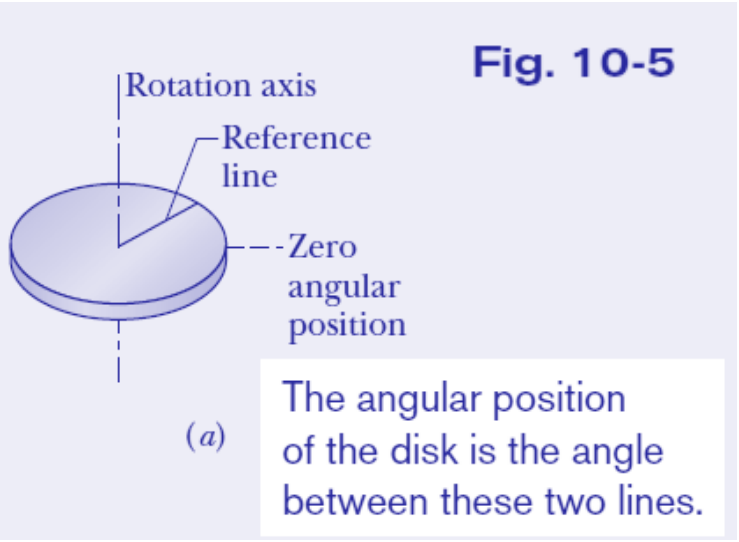
- The instantaneous angular acceleration a, is the limit of this quantity as Δt approaches zero.

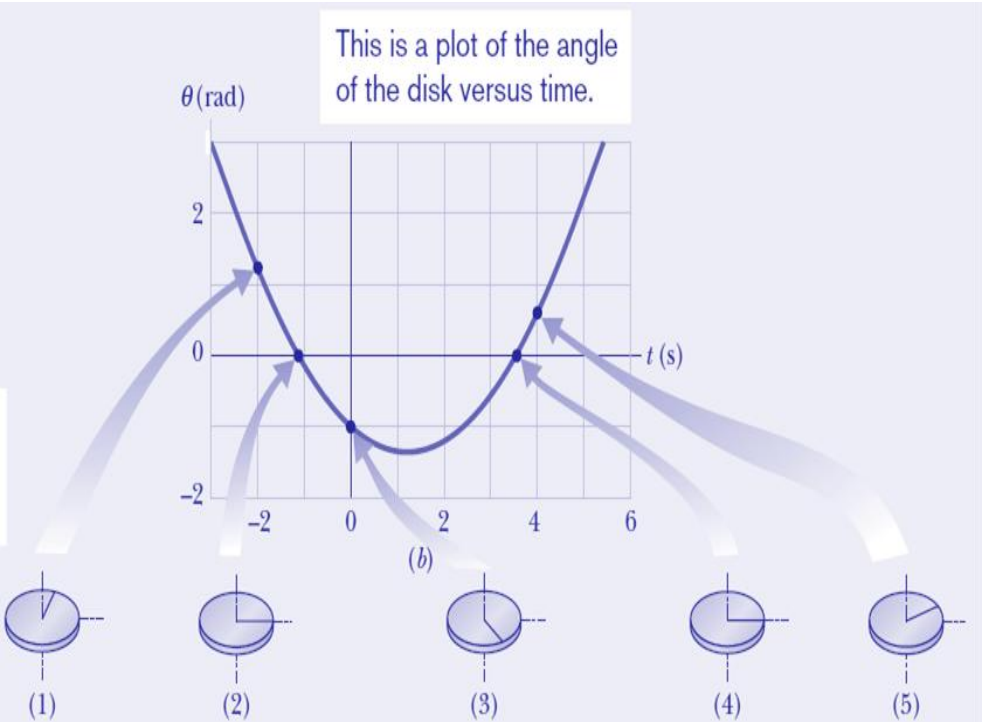
$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

- These relations hold for every particle of that body. The unit of angular acceleration is (rad/s^2).



Example





The disk in Fig. 10-5*a* is rotating about its central axis like a merry-go-round. The angular position $\theta(t)$ of a reference line on the disk is given by

$$\theta = -1.00 - 0.600t + 0.250t^2, \qquad (10\text{-}9)$$

with t in seconds, θ in radians, and the zero angular position as indicated in the figure.

(a) Graph the angular position of the disk versus time from $t = -3.0$ s to $t = 5.4$ s. Sketch the disk and its angular position reference line at $t = -2.0$ s, 0 s, and 4.0 s, and when the curve crosses the t axis.

Calculations: To sketch the disk and its reference line at a particular time, we need to determine θ for that time. To do so, we substitute the time into Eq. 10-9. For $t = -2.0$ s, we get

$$\theta = -1.00 - (0.600)(-2.0) + (0.250)(-2.0)^2$$

$$= 1.2 \text{ rad} = 1.2 \text{ rad} \frac{360^\circ}{2\pi \text{ rad}} = 69^\circ.$$

This means that at $t = -2.0$ s the reference line on the disk is rotated counterclockwise from the zero position by 1.2 rad $= 69^\circ$ (counterclockwise because θ is positive). Sketch 1 in Fig. 10-5*b* shows this position of the reference line.

Similarly, for $t = 0$, we find $\theta = -1.00$ rad $= -57^\circ$, which means that the reference line is rotated clockwise from the zero angular position by 1.0 rad, or 57°, as shown in sketch 3. For $t = 4.0$ s, we find $\theta = 0.60$ rad $= 34^\circ$ (sketch 5). Drawing sketches for when the curve crosses the t axis is easy, because then $\theta = 0$ and the reference line is momentarily aligned with the zero angular position (sketches 2 and 4).

Example

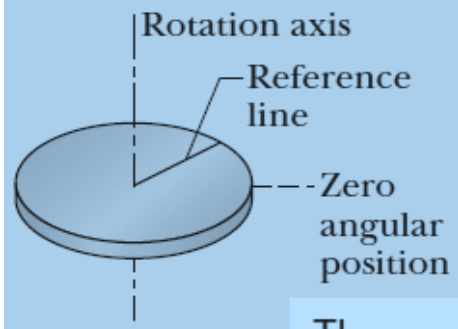
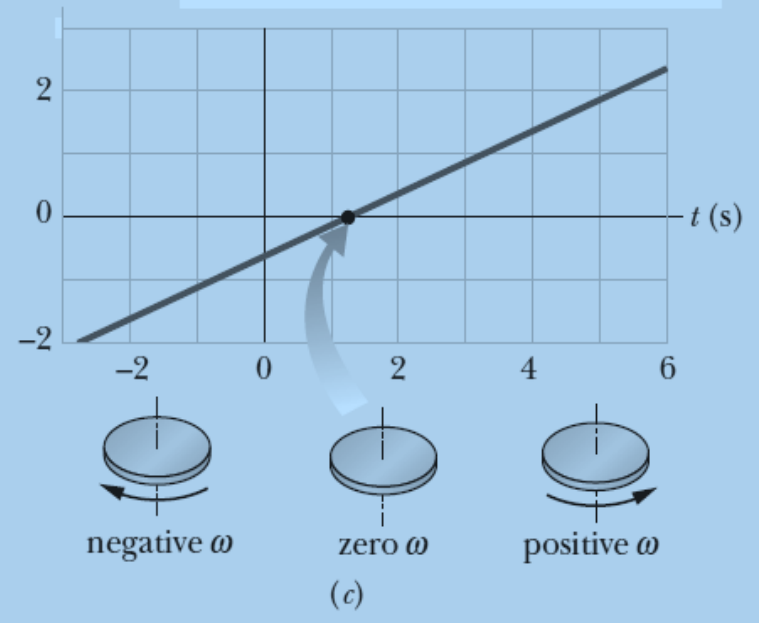


Fig. 10-5

The angular position of the disk is the angle between these two lines.

This is a plot of the angular velocity of the disk versus time.



(b) At what time t_{min} does $\theta(t)$ reach the minimum value shown in Fig. 10-5*b*? What is that minimum value?

Calculations: The first derivative of $\theta(t)$ is

$$\frac{d\theta}{dt} = -0.600 + 0.500t. \qquad (10\text{-}10)$$

Setting this to zero and solving for t give us the time at which $\theta(t)$ is minimum:

$$t_{min} = 1.20 \text{ s}. \qquad \text{(Answer)}$$

To get the minimum value of θ, we next substitute t_{min} into Eq. 10-9, finding

$$\theta = -1.36 \text{ rad} \approx -77.9°. \qquad \text{(Answer)}$$

Example

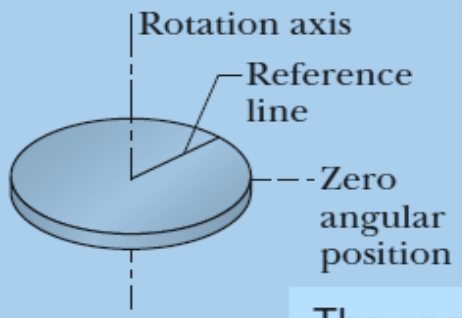


Fig. 10-5

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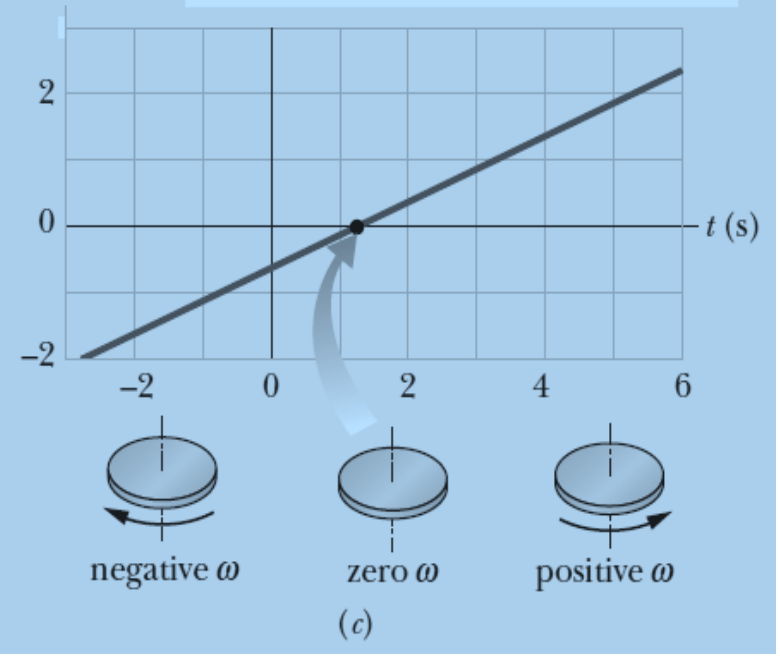


(c) Graph the angular velocity ω of the disk versus time from $t = -3.0$ s to $t = 6.0$ s. Sketch the disk and indicate the direction of turning and the sign of ω at $t = -2.0$ s, 4.0 s, and t_{min}.

Calculations: To sketch the disk at $t = -2.0$ s, we substitute that value into Eq. 10-11, obtaining

$$\omega = -1.6 \text{ rad/s.} \qquad \text{(Answer)}$$

The minus sign here tells us that at $t = -2.0$ s, the disk is turning clockwise (the left-hand sketch in Fig. 10-5*c*).

Substituting $t = 4.0$ s into Eq. 10-11 gives us

$$\omega = 1.4 \text{ rad/s.} \qquad \text{(Answer)}$$

The implied plus sign tells us that now the disk is turning counterclockwise (the righthand sketch in Fig. 10-5*c*).

For t_{min}, we already know that $d\theta/dt = 0$. So, we must also have $\omega = 0$. That is, the disk momentarily stops when the reference line reaches the minimum value of θ in Fig. 10-5*b*, as suggested by the center sketch in Fig. 10-5*c*. On the graph, this momentary stop is the zero point where the plot changes from the negative clockwise motion to the positive counterclockwise motion.

(d) Use the results in parts (a) through (c) to describe the motion of the disk from $t = -3.0$ s to $t = 6.0$ s.

Description: When we first observe the disk at $t = -3.0$ s, it has a positive angular position and is turning clockwise but slowing. It stops at angular position $\theta = -1.36$ rad and then begins to turn counterclockwise, with its angular position eventually becoming positive again.

Example: Angular Velocity and Acceleration

A child's top is spun with angular acceleration

$$\alpha = 5t^3 - 4t,$$

with t in seconds and α in radians per second-squared. At $t = 0$, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad.

(a) Obtain an expression for the angular velocity $\omega(t)$ of the top. That is, find an expression that explicitly indicates how the angular velocity depends on time. (We can tell that there *is* such a dependence because the top is undergoing an angular acceleration, which means that its angular velocity *is* changing.)

Calculations: Equation 10-8 tells us

$$d\omega = \alpha\, dt,$$

so

$$\int d\omega = \int \alpha\, dt.$$

From this we find

$$\omega = \int (5t^3 - 4t)\, dt = \tfrac{5}{4}t^4 - \tfrac{4}{2}t^2 + C.$$

To evaluate the constant of integration C, we note that $\omega = 5$ rad/s at $t = 0$. Substituting these values in our expression for ω yields

$$5 \text{ rad/s} = 0 - 0 + C,$$

so $C = 5$ rad/s. Then

$$\omega = \tfrac{5}{4}t^4 - 2t^2 + 5. \qquad \text{(Answer)}$$

(b) Obtain an expression for the angular position $\theta(t)$ of the top.

Calculations: Since Eq. 10-6 tells us that

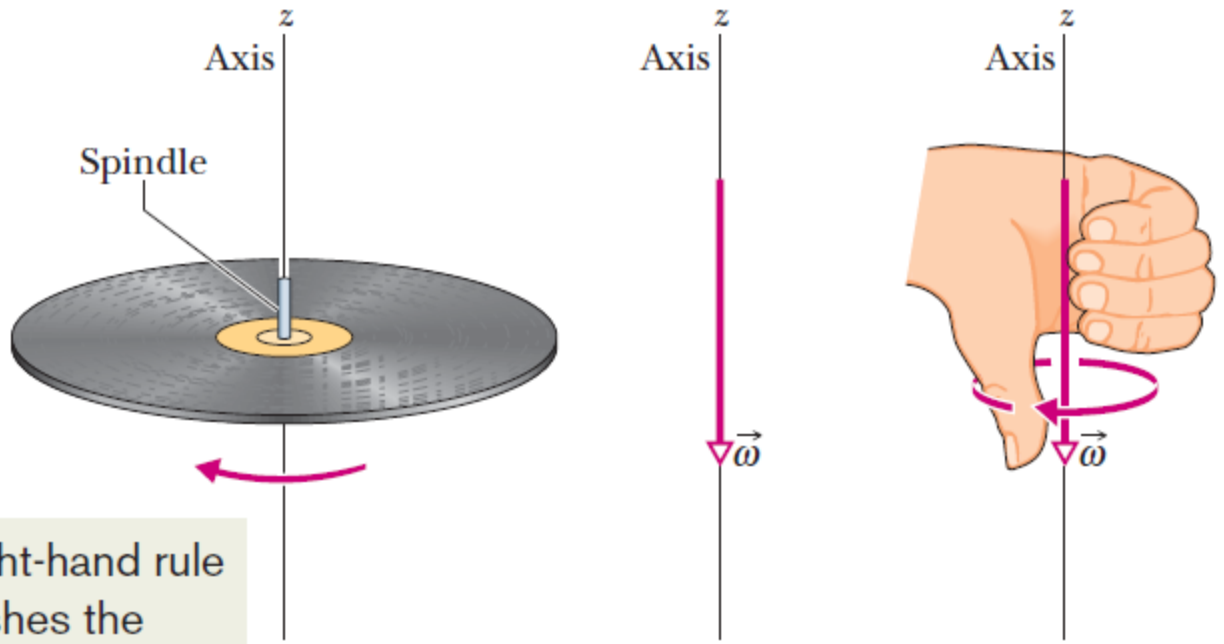
$$d\theta = \omega\, dt,$$

we can write

$$\theta = \int \omega\, dt = \int (\tfrac{5}{4}t^4 - 2t^2 + 5)\, dt$$
$$= \tfrac{1}{4}t^5 - \tfrac{2}{3}t^3 + 5t + C'$$
$$= \tfrac{1}{4}t^5 - \tfrac{2}{3}t^3 + 5t + 2, \qquad \text{(Answer)}$$

where C' has been evaluated by noting that $\theta = 2$ rad at $t = 0$.

10.3: Are Angular Quantities Vectors?



This right-hand rule establishes the direction of the angular velocity vector.

Fig. 10-6 (*a*) A record rotating about a vertical axis that coincides with the axis of the spindle. (*b*) The angular velocity of the rotating record can be represented by the vector $\vec{\omega}$, lying along the axis and pointing down, as shown. (*c*) We establish the direction of the angular velocity vector as downward by using a right-hand rule. When the fingers of the right hand curl around the record and point the way it is moving, the extended thumb points in the direction of $\vec{\omega}$.

10.4: Rotation with Constant Angular Acceleration

TABLE 10-1

Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

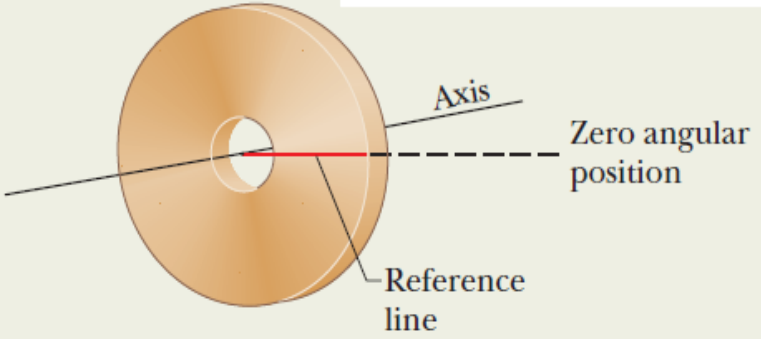
Equation Number	Linear Equation	Missing Variable		Angular Equation
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Just as in the basic equations for constant linear acceleration, the basic equations for constant angular acceleration can be derived in a similar manner. The constant angular acceleration equations are similar to the constant linear acceleration equations.

Example: Constant Angular Acceleration

Fig. 10-8

We measure rotation by using this reference line.
Clockwise = negative
Counterclockwise = positive



A grindstone (Fig. 10-8) rotates at constant angular acceleration $\alpha = 0.35\ \text{rad/s}^2$. At time $t = 0$, it has an angular velocity of $\omega_0 = -4.6$ rad/s and a reference line on it is horizontal, at the angular position $\theta_0 = 0$.

(a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0$ rev?

The angular acceleration is constant, so we can use the rotation equation:

$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2,$$

Substituting known values and setting $\theta_0 = 0$ and $\theta = 5.0$ rev $= 10\pi$ rad give us

$$10\pi\ \text{rad} = (-4.6\ \text{rad/s})t + \tfrac{1}{2}(0.35\ \text{rad/s}^2)t^2.$$

Solving this quadratic equation for *t, we find t =32 s.*

(b) Describe the grindstone's rotation between $t = 0$ and $t = 32$ s.

Description: The wheel is initially rotating in the negative (clockwise) direction with angular velocity $\omega_0 = 4.6$ rad/s, but its angular acceleration a is positive.

The initial opposite signs of angular velocity and angular acceleration means that the wheel slows in its rotation in the negative direction, stops, and then reverses to rotate in the positive direction.

After the reference line comes back through its initial orientation of $\theta = 0$, the wheel turns an additional 5.0 rev by time $t = 32$ s.

(c) At what time *t* does the grindstone momentarily stop?

Calculation: With $\omega = 0$, we solve for the corresponding time *t*.

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - (-4.6\ \text{rad/s})}{0.35\ \text{rad/s}^2} = 13\ \text{s}.$$

Example: Constant Angular Acceleration

Constant angular acceleration, riding a Rotor

While you are operating a Rotor (a large, vertical, rotating cylinder found in amusement parks), you spot a passenger in acute distress and decrease the angular velocity of the cylinder from 3.40 rad/s to 2.00 rad/s in 20.0 rev, at constant angular acceleration. (The passenger is obviously more of a "translation person" than a "rotation person.")

(a) What is the constant angular acceleration during this decrease in angular speed?

KEY IDEA

Because the cylinder's angular acceleration is constant, we can relate it to the angular velocity and angular displacement via the basic equations for constant angular acceleration (Eqs. 10-12 and 10-13).

Calculations: The initial angular velocity is $\omega_0 = 3.40$ rad/s, the angular displacement is $\theta - \theta_0 = 20.0$ rev, and the angular velocity at the end of that displacement is $\omega = 2.00$ rad/s. But we do not know the angular acceleration α and time t, which are in both basic equations.

To eliminate the unknown t, we use Eq. 10-12 to write

$$t = \frac{\omega - \omega_0}{\alpha},$$

which we then substitute into Eq. 10-13 to write

$$\theta - \theta_0 = \omega_0\left(\frac{\omega - \omega_0}{\alpha}\right) + \tfrac{1}{2}\alpha\left(\frac{\omega - \omega_0}{\alpha}\right)^2.$$

Solving for α, substituting known data, and converting 20 rev to 125.7 rad, we find

$$\alpha = \frac{\omega^2 - \omega_0^2}{2(\theta - \theta_0)} = \frac{(2.00\ \text{rad/s})^2 - (3.40\ \text{rad/s})^2}{2(125.7\ \text{rad})}$$

$$= -0.0301\ \text{rad/s}^2. \qquad \text{(Answer)}$$

(b) How much time did the speed decrease take?

Calculation: Now that we know α, we can use Eq. 10-12 to solve for t:

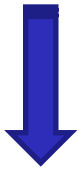
$$t = \frac{\omega - \omega_0}{\alpha} = \frac{2.00\ \text{rad/s} - 3.40\ \text{rad/s}}{-0.0301\ \text{rad/s}^2}$$

$$= 46.5\ \text{s}. \qquad \text{(Answer)}$$

10.5: Relating Linear and Angular Variables

- If a reference line on a rigid body rotates through an angle q, a point within the body at a position r from the rotation axis moves a distance s along a circular arc, where s is given by:

$$s = \theta r \qquad \text{(radian measure).}$$



- Differentiating the above equation with respect to time -- with *r* held constant -- leads to:

$$v = \omega r \qquad \text{(radian measure)}$$

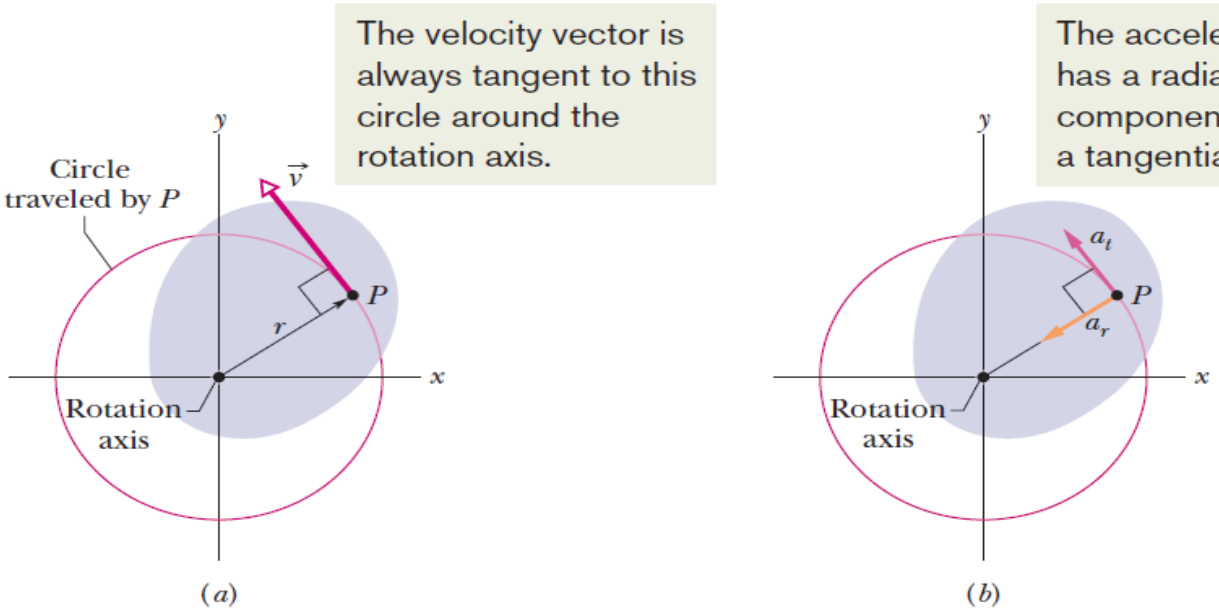
- The period of revolution *T* for the motion of each point and for the rigid body itself is given by:

$$T = \frac{2\pi r}{v}$$

- Substituting for v we find also that:

$$T = \frac{2\pi}{\omega} \qquad \text{(radian measure)}$$

10.5: Relating Linear and Angular Variables



Differentiating the velocity relation with respect to time—again with r held constant—leads to:

$$a_t = \alpha r \qquad \text{(radian measure)}$$

Here, $a_t = d\omega/dt$

Note that $dv/dt = a_t$,represents only the part of the linear acceleration that is responsible for changes in the magnitude **v** of the linear velocity. Like v, that part of the linear acceleration is tangent to the path of the point in question.

The radial part of the acceleration is the centripetal acceleration given by:

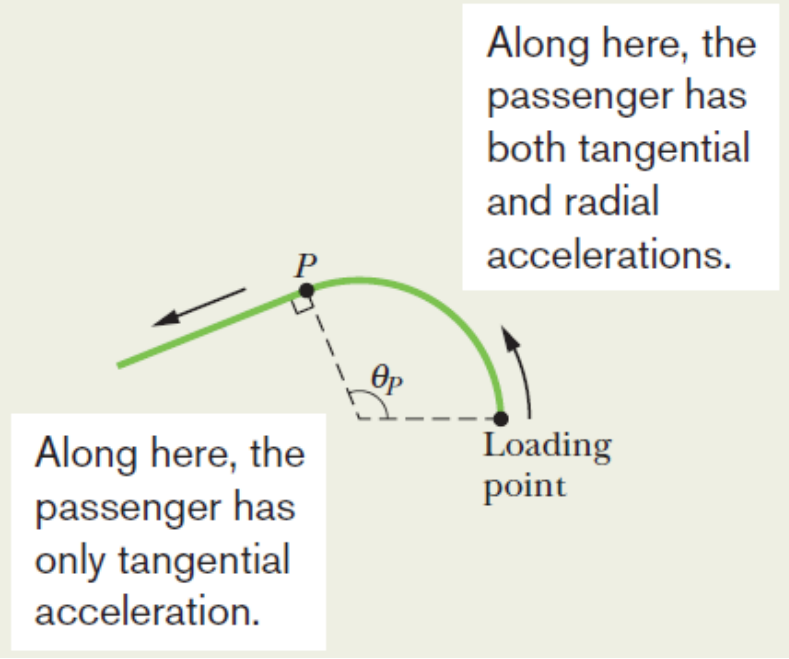
$$a_r = \frac{v^2}{r} = \omega^2 r$$

Example

Consider an induction roller coaster (which can be accelerated by magnetic forces even on a horizontal track). Each passenger is to leave the loading point with acceleration g along the horizontal track.

That first section of track forms a circular arc (Fig. 10-10), so that the passenger also experiences a centripetal acceleration. As the passenger accelerates along the arc, the magnitude of this centripetal acceleration increases alarmingly. When the magnitude a of the net acceleration reaches *4g* at some point P and angle Θ_P along the arc, the passenger moves in a straight line, along a tangent to the arc.

(a) What angle Θ_P should the arc subtend so that $a = 4g$ at point P?



Calculations:

Substituting $\omega_o = 0$, and $\Theta_o = 0$, and we find:

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0).$$

$$\omega^2 = \frac{2a_t\theta}{r}.$$

But,

$$a_r = \omega^2 r$$

which gives:

$$a_r = 2a_t\theta.$$

This leads us to a total acceleration:

$$a = \sqrt{a_t^2 + a_r^2}.$$

Substituting for a_r, and solving for Θ leads to:

$$\theta = \frac{1}{2}\sqrt{\frac{a^2}{a_t^2} - 1}$$

When a reaches the design value of *4g*, angle is the angle Θ_P. Substituting $a = 4g$, $\Theta = \Theta_P$, and $a_t = g$, we find:

$$\theta_P = \frac{1}{2}\sqrt{\frac{(4g)^2}{g^2} - 1} = 1.94\ \text{rad} = 111°.$$

Example, cont.

(b) What is the magnitude *a* of the passenger's net acceleration at point P and after point P?

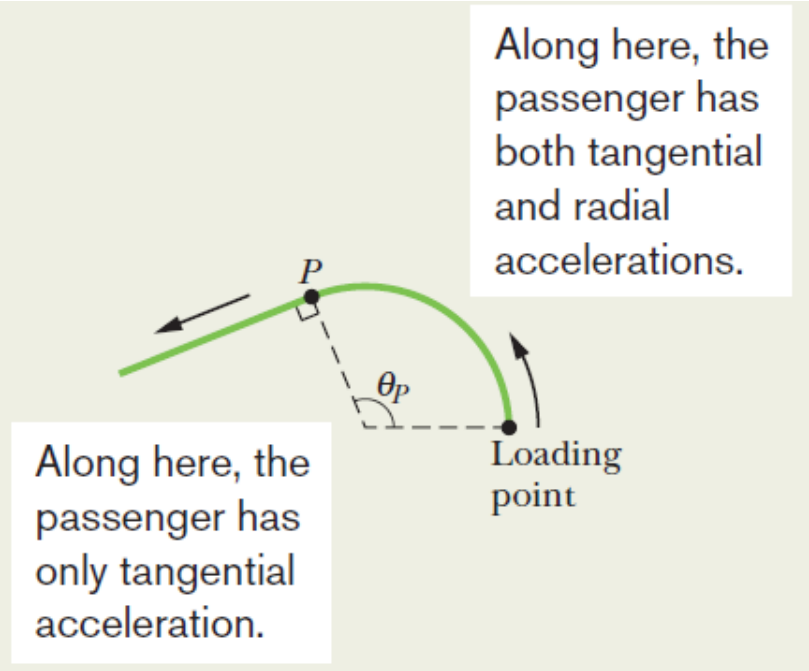
Reasoning: At *P*, a has the design value of *4g*. Just after *P* is reached, the passenger moves in a straight line and no longer has centripetal acceleration.

Thus, the passenger has only the acceleration magnitude g along the track.

Hence, $a = 4$g at P and $a = g$ after *P*. (Answer)

Roller-coaster headache can occur when a passenger's head undergoes an abrupt change in acceleration, with the acceleration magnitude large before or after the change.

The reason is that the change can cause the brain to move relative to the skull, tearing the veins that bridge the brain and skull. Our design to increase the acceleration from g to 4g along the path to P might harm the passenger, but the abrupt change in acceleration as the passenger passes through point P is more likely to cause roller-coaster headache.



10.6: Kinetic Energy of Rotation

- For an extended rotating rigid body, treat the body as a collection of particles with different speeds, and add up the kinetic energies of all the particles to find the total kinetic energy of the body:

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 + \cdots = \sum \tfrac{1}{2}m_iv_i^2,$$

(m_i is the mass of the i^{th} particle and v_i is its speed).

$$K = \sum \tfrac{1}{2}m_i(\omega r_i)^2 = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2,$$ (ω is the same for all particles)

- The quantity in parentheses on the right side is called the rotational inertia (or moment of inertia) I of the body with respect to the axis of rotation. It is a constant for a particular rigid body and a particular rotation axis which must always be specified.

- Therefore,

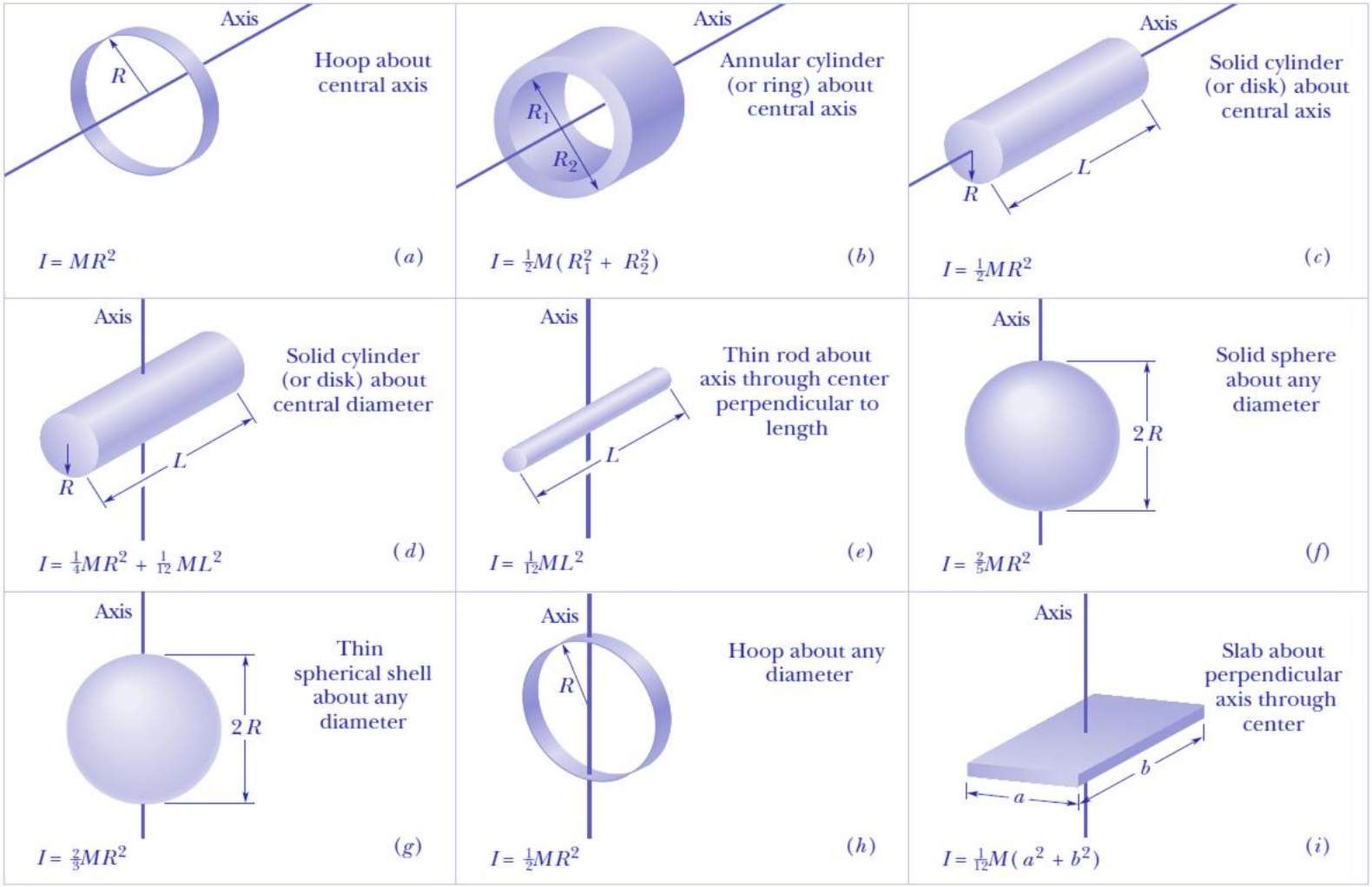
$$I = \sum m_i r_i^2 \quad \text{(rotational inertia)} \qquad \Rightarrow \qquad K = \tfrac{1}{2}I\omega^2 \quad \text{(radian measure)}$$

10.7: Calculating the Rotational Inertia

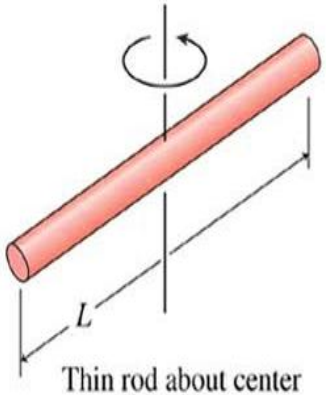
If a rigid body consists of a great many adjacent particles (it is continuous, like a Frisbee), we consider an integral and define the rotational inertia of the body as:

$$I = \int r^2\, dm \qquad \text{(rotational inertia, continuous body).}$$

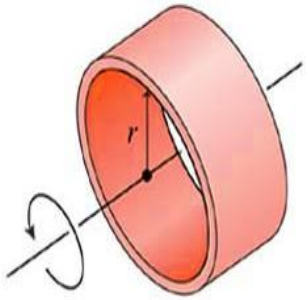
Some Rotational Inertias



10.7: Calculating the Rotational Inertia

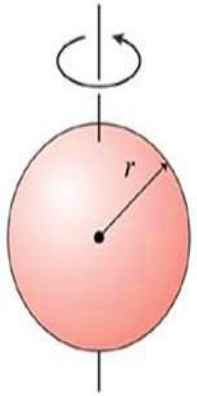


Thin rod about center
$I = \frac{1}{12}ML^2$

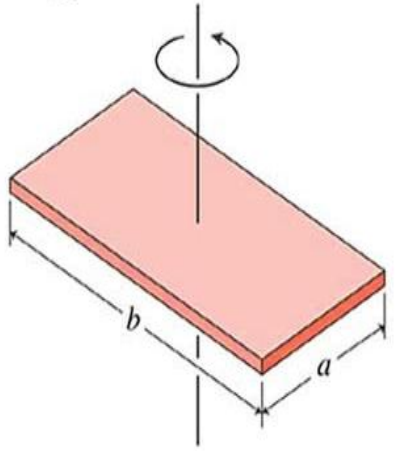


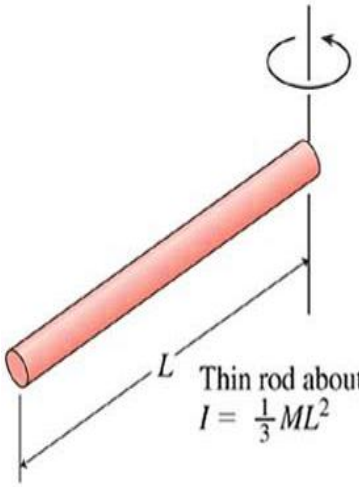
Thin ring or hollow cylinder about its axis
$I = MR^2$

Solid sphere about diameter
$I = \frac{2}{5}MR^2$

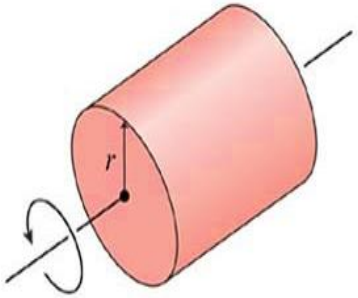


Flat plate about perpendicular axis
$I = \frac{1}{12}M(a^2 + b^2)$



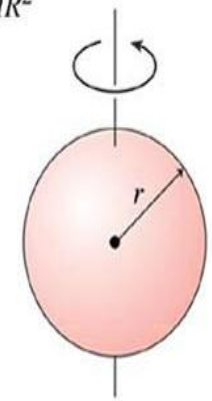


Thin rod about end
$I = \frac{1}{3}ML^2$

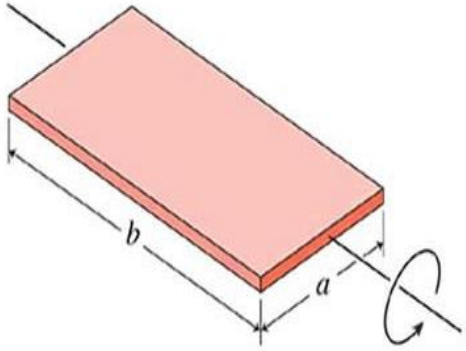


Disk or solid cylinder about its axis
$I = \frac{1}{2}MR^2$

Hollow spherical shell about diameter
$I = \frac{2}{3}MR^2$



Flat plate about central axis
$I = \frac{1}{12}Ma^2$

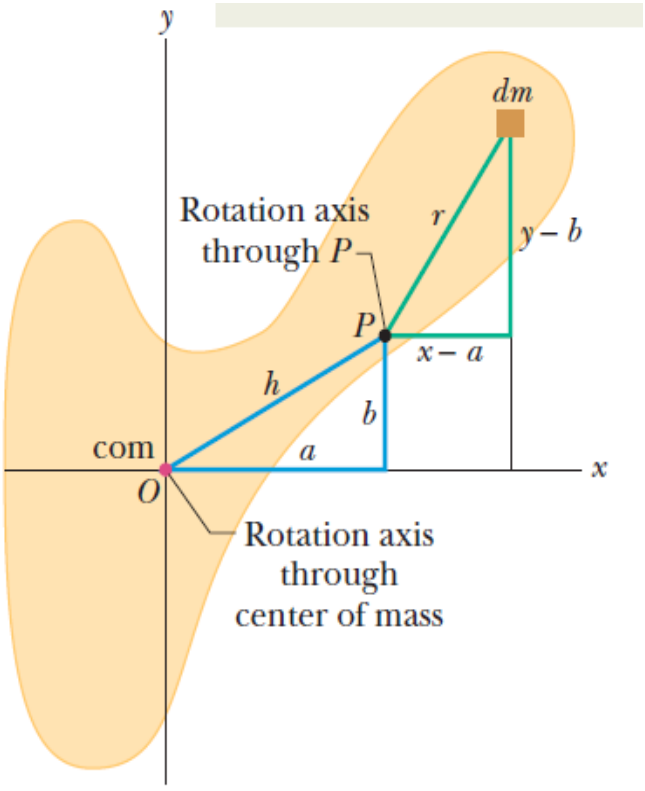


10.7: Calculating the Rotational Inertia

Parallel Axis Theorem:

If *h* is a perpendicular distance between a given axis and the axis through the center of mass (these two axes being parallel).Then the rotational inertia I about the given axis is:

$$I = I_{com} + Mh^2 \qquad \text{(parallel-axis theorem)}$$



• Let O be the center of mass (and also the origin of the coordinate system) of the arbitrarily shaped body shown in cross section.
• Consider an axis through O perpendicular to the plane of the figure, and another axis through point P parallel to the first axis.
• Let the *x* and *y* coordinates of P be a and b.
• Let *dm* be a mass element with the general coordinates x and y. The rotational inertia of the body about the axis through P is:

$$I = \int r^2\, dm = \int [(x-a)^2 + (y-b)^2]\, dm$$

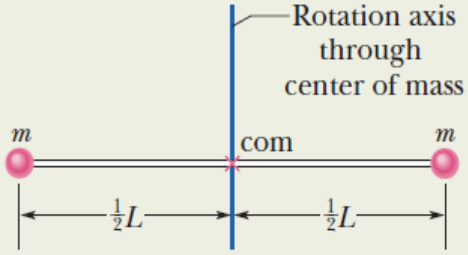
$$= \int (x^2 + y^2)\, dm - 2a \int x\, dm - 2b \int y\, dm + \int (a^2 + b^2)\, dm$$

• But $x^2 + y^2 = R^2$, where *R* is the distance from *O* to *dm*, the first integral is simply I_{com}, the rotational inertia of the body about an axis through its center of mass.
• The last term in is Mh^2, where *M* is the body's total mass.

Example: Rotational Inertia

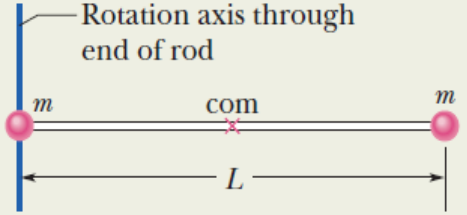
Figure 10-13*a* shows a rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.

(a) What is the rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod as shown?



(*a*)

Here the rotation axis is through the com.



(*b*)

Here it has been shifted from the com without changing the orientation. We can use the parallel-axis theorem.

Calculations: For the two particles, each at perpendicular distance $\frac{1}{2}L$ from the rotation axis, we have

$$I = \sum m_i r_i^2 = (m)(\tfrac{1}{2}L)^2 + (m)(\tfrac{1}{2}L)^2$$
$$= \tfrac{1}{2}mL^2. \qquad \text{(Answer)}$$

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis (Fig. 10-13*b*)?

First technique: We calculate I as in part (a), except here the perpendicular distance r_i is zero for the particle on the left and L for the particle on the right. Now Eq. 10-33 gives us

$$I = m(0)^2 + mL^2 = mL^2. \qquad \text{(Answer)}$$

Second technique: Because we already know I_{com} about an axis through the center of mass and because the axis here is parallel to that "com axis," we can apply the parallel-axis theorem (Eq. 10-36). We find

$$I = I_{com} + Mh^2 = \tfrac{1}{2}mL^2 + (2m)(\tfrac{1}{2}L)^2$$
$$= mL^2. \qquad \text{(Answer)}$$

Example: Rotational Inertia

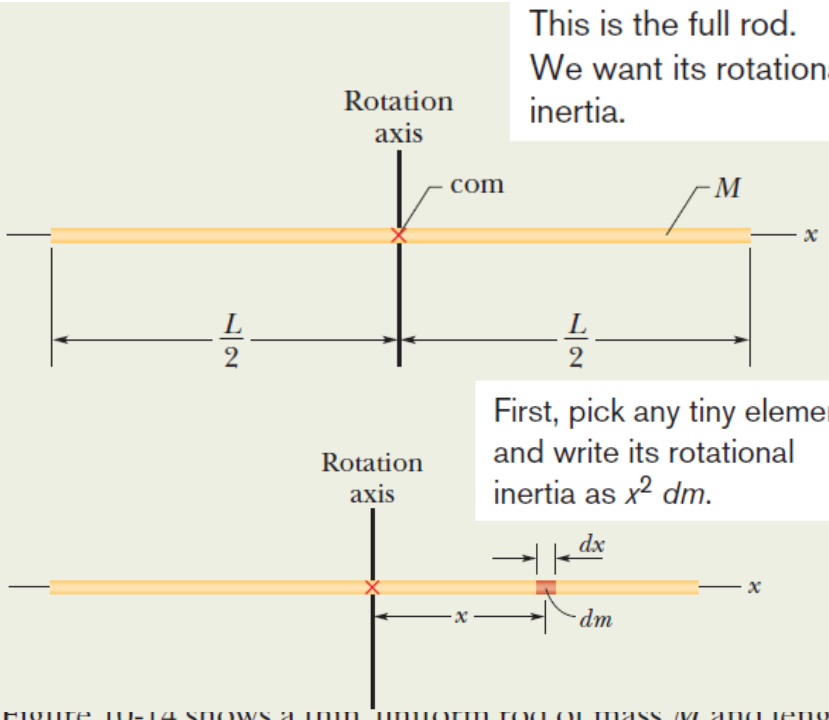


Figure 10-14 shows a thin, uniform rod of mass M and length L, on an x axis with the origin at the rod's center.

(a) What is the rotational inertia of the rod about the perpendicular rotation axis through the center?

KEY IDEAS

(1) Because the rod is uniform, its center of mass is at its center. Therefore, we are looking for I_{com}. (2) Because the rod is a continuous object, we must use the integral of Eq. 10-35,

$$I = \int r^2\,dm, \qquad (10\text{-}38)$$

to find the rotational inertia.

Calculations: We want to integrate with respect to coordinate x (not mass m as indicated in the integral), so we must relate the mass dm of an element of the rod to its length dx along the rod. (Such an element is shown in Fig. 10-14.) Because the rod is uniform, the ratio of mass to length is the same for all the elements and for the rod as a whole. Thus, we can write

$$\frac{\text{element's mass } dm}{\text{element's length } dx} = \frac{\text{rod's mass } M}{\text{rod's length } L}$$

or $$dm = \frac{M}{L}\,dx.$$

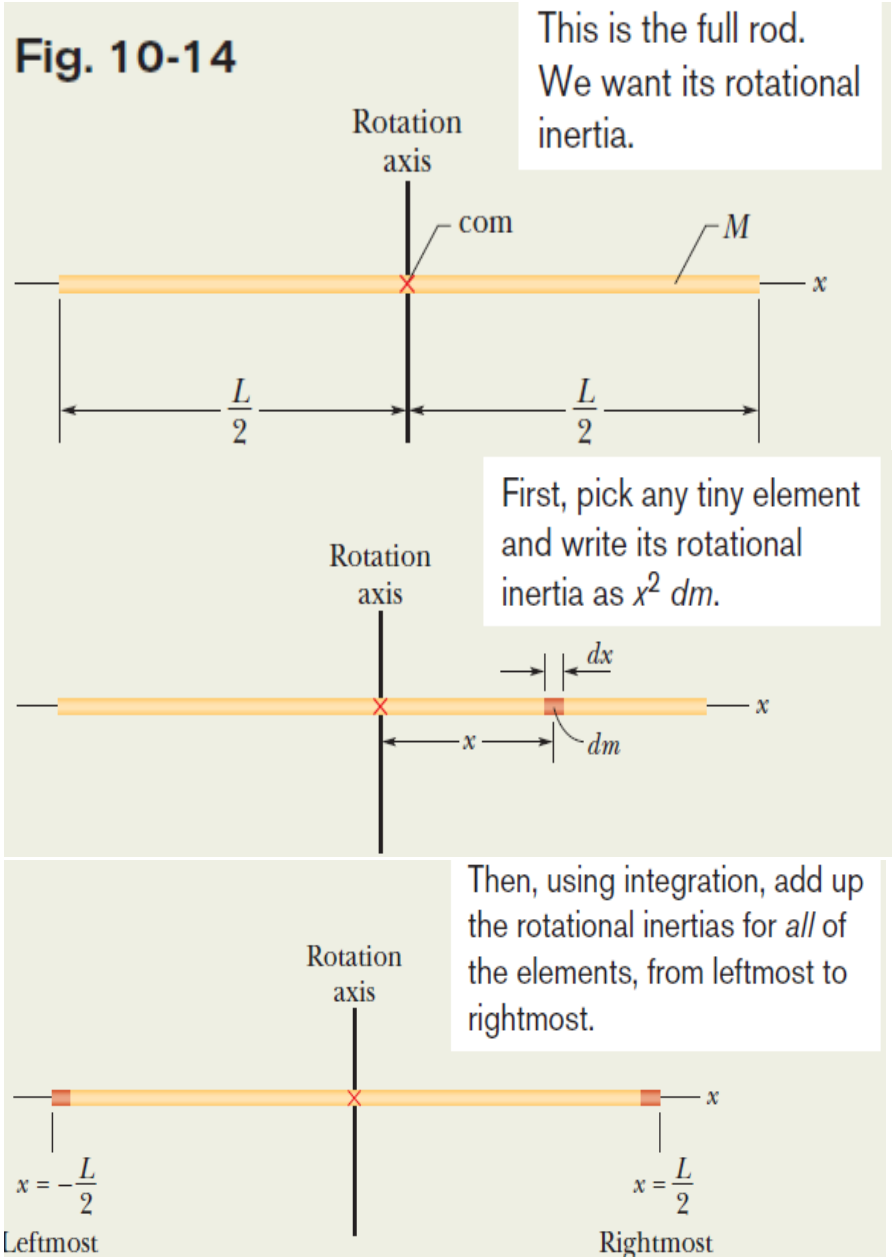
We can now substitute this result for dm and x for r in Eq. 10-38. Then we integrate from end to end of the rod (from $x = -L/2$ to $x = L/2$) to include all the elements. We find

$$I = \int_{x=-L/2}^{x=+L/2} x^2 \left(\frac{M}{L}\right) dx$$

$$= \frac{M}{3L}\left[x^3\right]_{-L/2}^{+L/2} = \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right]$$

$$= \tfrac{1}{12}ML^2. \qquad \text{(Answer)}$$

Example: Rotational Inertia



(b) What is the rod's rotational inertia I about a new rotation axis that is perpendicular to the rod and through the left end?

KEY IDEAS

We can find I by shifting the origin of the x axis to the left end of the rod and then integrating from $x = 0$ to $x = L$. However, here we shall use a more powerful (and easier) technique by applying the parallel-axis theorem (Eq. 10-36), in which we shift the rotation axis without changing its orientation.

Calculations: If we place the axis at the rod's end so that it is parallel to the axis through the center of mass, then we can use the parallel-axis theorem (Eq. 10-36). We know from part (a) that I_{com} is $\frac{1}{12}ML^2$. From Fig. 10-14, the perpendicular distance h between the new rotation axis and the center of mass is $\frac{1}{2}L$. Equation 10-36 then gives us

$$I = I_{com} + Mh^2 = \tfrac{1}{12}ML^2 + (M)(\tfrac{1}{2}L)^2$$
$$= \tfrac{1}{3}ML^2. \qquad \text{(Answer)}$$

Actually, this result holds for any axis through the left or right end that is perpendicular to the rod, whether it is parallel to the axis shown in Fig. 10-14 or not.

Example: Rotational KE

Large machine components that undergo prolonged, high-speed rotation are first examined for the possibility of failure in a *spin test system.* In this system, a component is *spun up* (brought up to high speed) while inside a cylindrical arrangement of lead bricks and containment liner, all within a steel shell that is closed by a lid clamped into place. If the rotation causes the component to shatter, the soft lead bricks are supposed to catch the pieces for later analysis.

In 1985, Test Devices, Inc. (www.testdevices.com) was spin testing a sample of a solid steel rotor (a disk) of mass $M = 272$ kg and radius $R = 38.0$ cm. When the sample reached an angular speed ω of 14 000 rev/min, the test engineers heard a dull thump from the test system, which was located one floor down and one room over from them. Investigating, they found that lead bricks had been thrown out in the hallway leading to the test room, a door to the room had been hurled into the adjacent parking lot, one lead brick had shot from the test site through the wall of a neighbor's kitchen, the structural beams of the test building had been damaged, the concrete floor beneath the spin chamber had been shoved downward by about 0.5 cm, and the 900 kg lid had been blown upward through the ceiling and had then crashed back onto the test equipment (Fig. 10-15). The exploding pieces had not penetrated the room of the test engineers only by luck.

How much energy was released in the explosion of the rotor?

KEY IDEA

The released energy was equal to the rotational kinetic energy K of the rotor just as it reached the angular speed of 14 000 rev/min.

Fig. 10-15 Some of the destruction caused by the explosion of a rapidly rotating steel disk. *(Courtesy Test Devices, Inc.)*

Calculations: We can find K with Eq. 10-34 ($K = \frac{1}{2}I\omega^2$), but first we need an expression for the rotational inertia I. Because the rotor was a disk that rotated like a merry-go-round, I is given by the expression in Table 10-2*c* ($I = \frac{1}{2}MR^2$). Thus, we have

$$I = \tfrac{1}{2}MR^2 = \tfrac{1}{2}(272\text{ kg})(0.38\text{ m})^2 = 19.64\text{ kg}\cdot\text{m}^2.$$

The angular speed of the rotor was

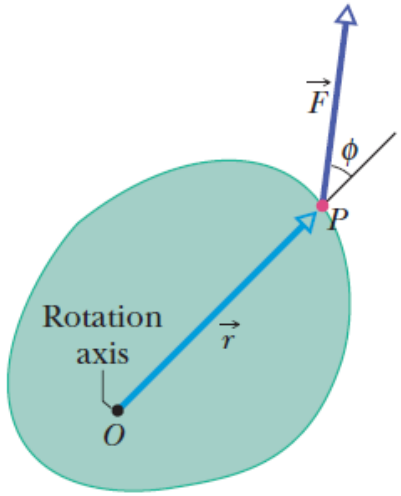
$$\omega = (14\,000\text{ rev/min})(2\pi\text{ rad/rev})\left(\frac{1\text{ min}}{60\text{ s}}\right)$$
$$= 1.466 \times 10^3\text{ rad/s}.$$

Now we can use Eq. 10-34 to write

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(19.64\text{ kg}\cdot\text{m}^2)(1.466 \times 10^3\text{ rad/s})^2$$
$$= 2.1 \times 10^7\text{ J}. \qquad \text{(Answer)}$$

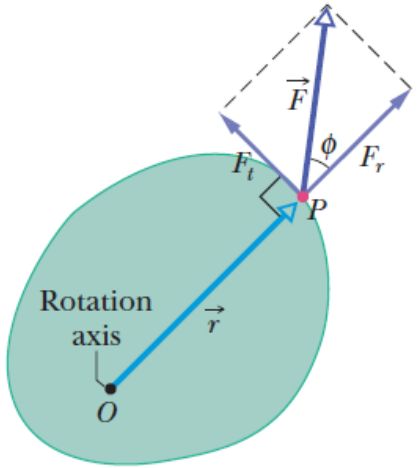
Being near this explosion was quite dangerous.

10.8: Torque



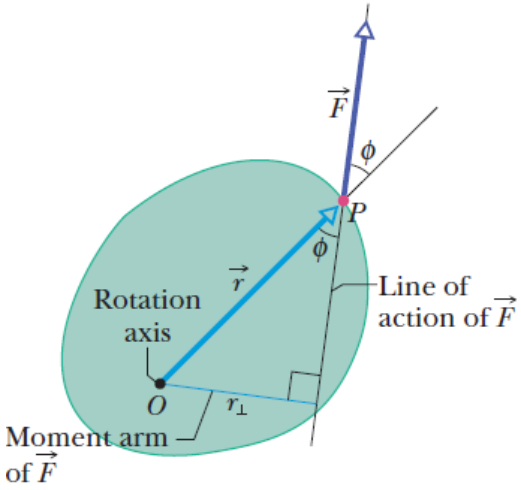
The torque due to this force causes rotation around this axis (which extends out toward you).

(*a*)



But actually only the *tangential* component of the force causes the rotation.

(*b*)



You calculate the same torque by using this moment arm distance and the full force magnitude.

(*c*)

The ability of a force *F* to rotate the body depends on both the magnitude of its tangential component F_t, and also on just how far from *O*, the pivot point, the force is applied.

To include both these factors, a quantity called **torque t** is defined as:

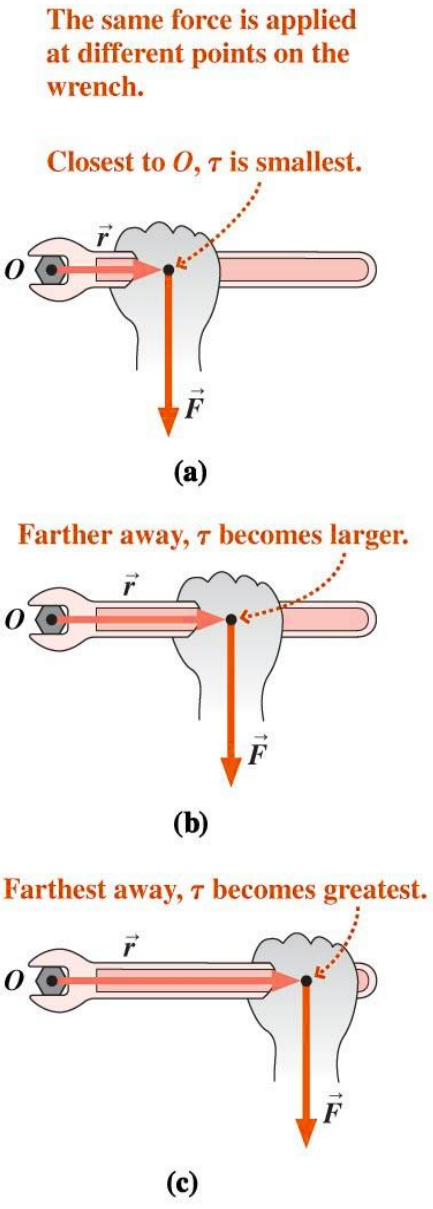
$$\tau = (r)(F \sin \phi).$$

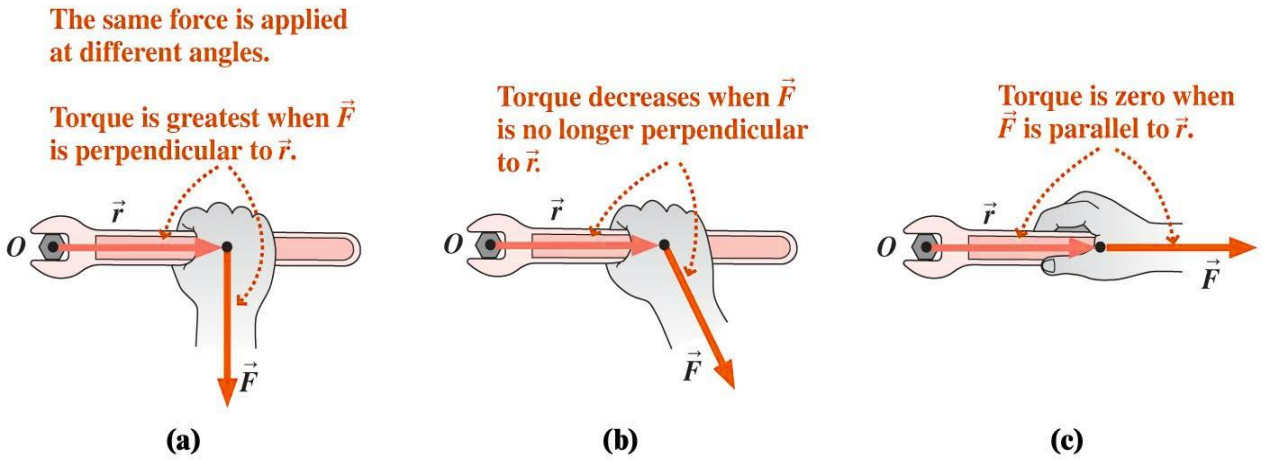
OR, $$\tau = (r)(F \sin \phi) = rF_t$$

where $r_\perp$ is called the moment arm of F. $$\tau = (r \sin \phi)(F) = r_\perp F,$$

10.8: Torque

- Torque τ is the rotational analog of force, and results from the application of one or more forces.
  - Torque is relative to a chosen rotation axis.
  - Torque depends on
    - The distance from the rotation axis to the force application point.
    - The magnitude of the force $\vec{F}$
    - The orientation of the force relative to the displacement $\vec{r}$ from axis to force application point: $\tau = rF\sin\theta$





Clicker question

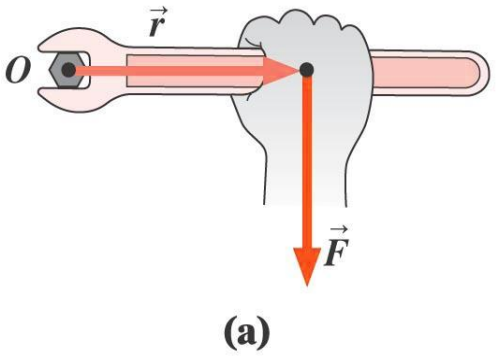
The forces in the figures all have the same magnitude. Which force produces zero torque?

A. The force in figure (a)

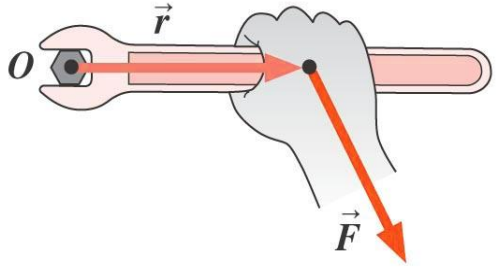
B. The force in figure (b)

C. The force in figure (c)

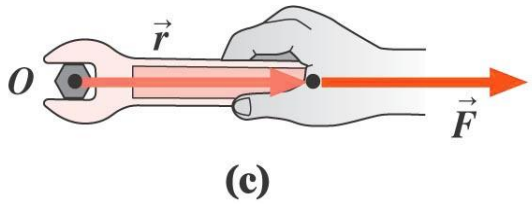
D. All of the forces produce torque



(a)



(b)



(c)

10.9: Newton's 2nd Law for Rotation

$$F_t = ma_t.$$

$$\tau = F_t r = ma_t r.$$

$$\tau = m(\alpha r)r = (mr^2)\alpha.$$

$$\tau = I\alpha$$

For more than one force, we can generalize:

$$\tau_{\text{net}} = I\alpha \qquad \text{(radian measure)}$$

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

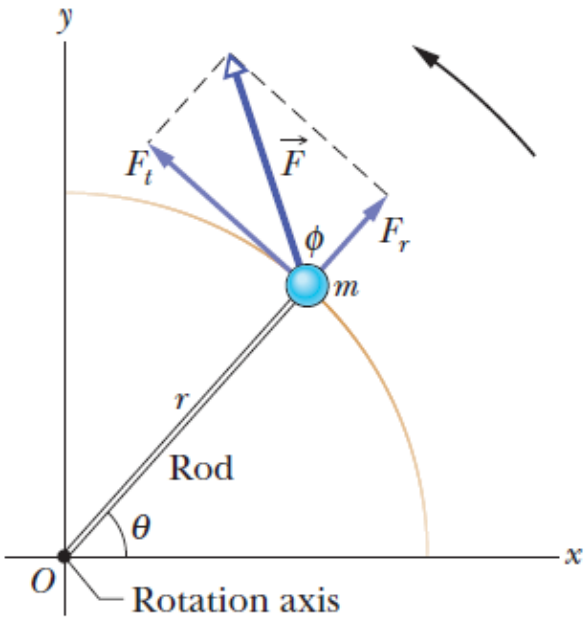
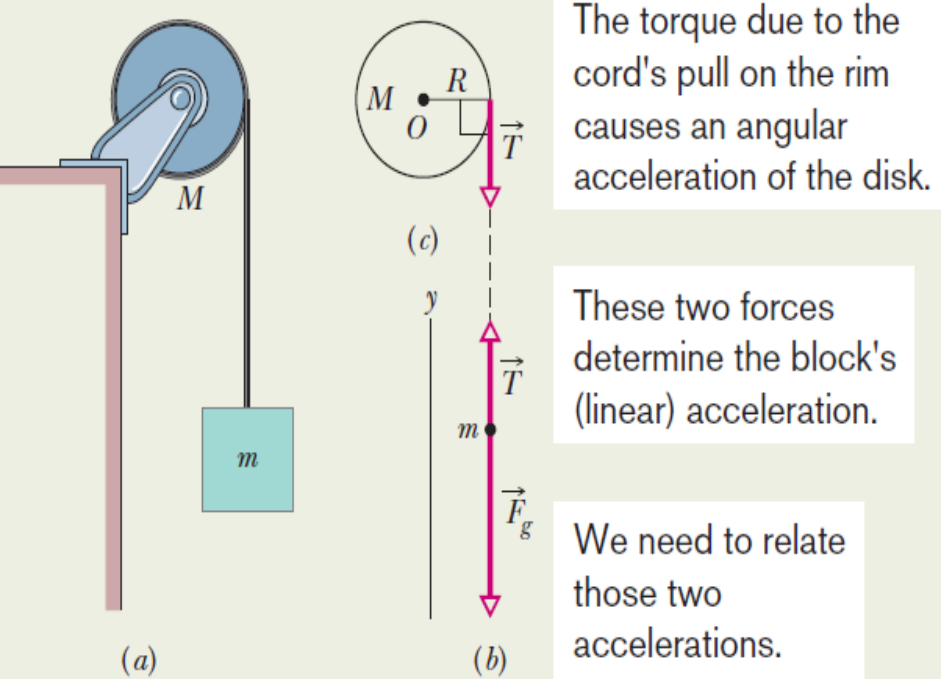


Fig. 10-17 A simple rigid body, free to rotate about an axis through O, consists of a particle of mass m fastened to the end of a rod of length r and negligible mass. An applied force $\vec{F}$ causes the body to rotate.

Example: Newton's 2nd Law in Rotational Motion

Figure 10-18*a* shows a uniform disk, with mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed horizontal axle. A block with mass $m = 1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.



The torque due to the cord's pull on the rim causes an angular acceleration of the disk.

These two forces determine the block's (linear) acceleration.

We need to relate those two accelerations.

Forces on block:

From the block's free body diagram, we can write Newton's second law for components along a vertical y axis as: T – mg = ma

The torque from the tension force, **T**, is -RT, negative because the torque rotates the disk clockwise from rest. The rotational inertia I of the disk is ½ MR^2. But $\Sigma\tau = Ia = -RT = 1/2\ MR^2a$.

Because the cord does not slip, the linear acceleration *a* of the block and the (tangential) linear acceleration a_t of the rim of the disk are equal.
We now have: T=-1/2 Ma.

Combining results:

We then find T:

$$a = -g\frac{2m}{M+2m} = -(9.8\ \text{m/s}^2)\frac{(2)(1.2\ \text{kg})}{2.5\ \text{kg} + (2)(1.2\ \text{kg})}$$
$$= -4.8\ \text{m/s}^2. \qquad \text{(Answer)}$$

The angular acceleration of the disk is:

$$T = -\tfrac{1}{2}Ma = -\tfrac{1}{2}(2.5\ \text{kg})(-4.8\ \text{m/s}^2)$$
$$= 6.0\ \text{N}. \qquad \text{(Answer)}$$

Note that the acceleration a of the falling block is less

$$\alpha = \frac{a}{R} = \frac{-4.8\ \text{m/s}^2}{0.20\ \text{m}} = -24\ \text{rad/s}^2. \qquad \text{(Answer)}$$

than g, and tension T in the cord (=6.0 N) is less than the gravitational force on the hanging block (mg =11.8 N).

10.10: Work and Rotational Kinetic Energy

$$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W \qquad \text{(work–kinetic energy theorem).}$$

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta \qquad \text{(work, rotation about fixed axis),}$$

where τ is the torque doing the work *W, and* Θ_i *and* Θ_f *are the body's angular* positions before and after the work is done, respectively. When τ is constant,

$$W = \tau(\theta_f - \theta_i) \qquad \text{(work, constant torque)}$$

The rate at which the work is done is the power,

$$P = \frac{dW}{dt} = \tau\omega \qquad \text{(power, rotation about fixed axis)}$$

10.10: Work and Rotational Kinetic Energy

TABLE 10-3

Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{\text{net}} = ma$	Newton's second law	$\tau_{\text{net}} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Example: Work, Rotational KE, Torque

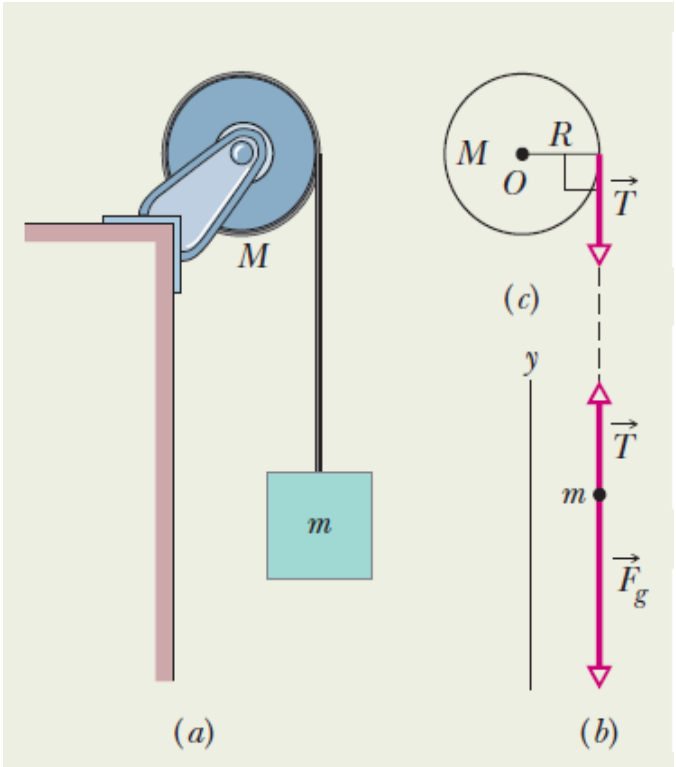
Let the disk in Fig. 10-18 start from rest at time $t = 0$ and also let the tension in the massless cord be 6.0 N and the angular acceleration of the disk be –24 rad/s^2. What is its rotational kinetic energy K at $t = 2.5$ s?

Calculations: Because we want ω and know α and ω_0 $(= 0)$, we use Eq. 10-12:

$$\omega = \omega_0 + \alpha t = 0 + \alpha t = \alpha t.$$

Substituting $\omega = \alpha t$ and $I = \frac{1}{2}MR^2$ into Eq. 10-34, we find

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(\tfrac{1}{2}MR^2)(\alpha t)^2 = \tfrac{1}{4}M(R\alpha t)^2$$
$$= \tfrac{1}{4}(2.5\text{ kg})[(0.20\text{ m})(-24\text{ rad/s}^2)(2.5\text{ s})]^2$$
$$= 90\text{ J}. \qquad \text{(Answer)}$$



We can also get this answer by finding the disk's kinetic energy from the work done on the disk.

Calculations: First, we relate the *change* in the kinetic energy of the disk to the net work W done on the disk, using the work–kinetic energy theorem of Eq. 10-52 $(K_f - K_i = W)$. With K substituted for K_f and 0 for K_i, we get

$$K = K_i + W = 0 + W = W. \qquad (10\text{-}60)$$

Next we want to find the work W. We can relate W to the torques acting on the disk with Eq. 10-53 or 10-54. The only torque causing angular acceleration and doing work is the torque due to force $\vec{T}$ on the disk from the cord, which is equal to $-TR$. Because α is constant, this torque also must be constant. Thus, we can use Eq. 10-54 to write

$$W = \tau(\theta_f - \theta_i) = -TR(\theta_f - \theta_i). \qquad (10\text{-}61)$$

Because α is constant, we can use Eq. 10-13 to find $\theta_f - \theta_i$. With $\omega_i = 0$, we have

$$\theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2 = 0 + \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}\alpha t^2.$$

Now we substitute this into Eq. 10-61 and then substitute the result into Eq. 10-60. Inserting the given values $T = 6.0$ N and $\alpha = -24$ rad/s^2, we have

$$K = W = -TR(\theta_f - \theta_i) = -TR(\tfrac{1}{2}\alpha t^2) = -\tfrac{1}{2}TR\alpha t^2$$
$$= -\tfrac{1}{2}(6.0\text{ N})(0.20\text{ m})(-24\text{ rad/s}^2)(2.5\text{ s})^2$$
$$= 90\text{ J}. \qquad \text{(Answer)}$$